Exercise 94

Cobalt-60 has a half-life of 5.24 years.

(a) Find the mass that remains from a 100-mg sample after 20 years.

(b) How long would it take for the mass to decay to 1 mg?

Solution

Start with the assumption that the rate of mass decay is proportional to the mass.

$$\frac{dm}{dt} \propto -m$$

Change this proportionality to an equation by introducing a constant.

$$\frac{dm}{dt} = -km$$

Divide both sides by m.

$$\frac{1}{m}\frac{dm}{dt} = -k$$

Rewrite the left side as a derivative of a logarithm by using the chain rule.

$$\frac{d}{dt}\ln m = -k$$

The function you take a derivative of to get $-k$ is $-kt + C$, where C is any constant.

$$\ln m = -kt + C$$

Exponentiate both sides to solve for m.

$$e^{\ln m} = e^{-kt+C}$$

$$m(t) = e^C e^{-kt}$$

Use a new constant m_0 for e^C.

$$m(t) = m_0 e^{-kt}$$

Use the fact that the half-life of cobalt-60 is 5.24 years to determine k.

$$m(5.24) = m_0 e^{-k(5.24)}$$

$$\frac{m_0}{2} = m_0 e^{-5.24k}$$

$$\frac{1}{2} = e^{-5.24k}$$

$$\ln\frac{1}{2} = \ln e^{-5.24k}$$

$$-\ln 2 = (-5.24k)\ln e$$

$$k = \frac{\ln 2}{5.24} \approx 0.13228 \text{ year}^{-1}$$

As a result,

$$m(t) = m_0 e^{-\left(\frac{\ln 2}{5.24}\right)t}$$

$$= m_0 e^{\ln 2^{-t/5.24}}$$

$$= m_0 (2)^{-t/5.24}.$$

Part (a)

After 20 years, the mass of a 100-mg sample will be

$$m(20) = 100(2)^{-20/5.24} \approx 7.09628 \text{ mg}.$$

Part (b)

To find when the mass will be 1 mg, set $m(t) = 1$ and solve the equation for t.

$$m(t) = 1$$

$$100(2)^{-t/5.24} = 1$$

$$2^{-t/5.24} = \frac{1}{100}$$

$$\ln 2^{-t/5.24} = \ln \frac{1}{100}$$

$$\left(-\frac{t}{5.24}\right) \ln 2 = -\ln 100$$

$$t = \frac{5.24 \ln 100}{\ln 2} \approx 34.8138 \text{ years}$$